VI. THE $L = \hat{R} + \hat{R}^{\alpha}{}_{\beta\gamma\delta}\, \hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}$ THEORY

1. Introduction

In this chapter I investigate the theory of gravity based on the Lagrangian,

$$L = - s \frac{\hbar c}{16\pi L^2} \hat{R} - \frac{\hbar c}{16\pi\alpha_G} \hat{R}^{\alpha}{}_{\beta\gamma\delta}\, \hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta} + L_M. \qquad (1)$$

This theory has been proposed and investigated independently by Fairchild and by Mansouri and Chang. I begin checking out the criteria listed in the previous chapter. The process is far from complete and work is still in progress.

In Section 2, I start by writing out the field equations. They involve no higher than first derivatives of the frame and second derivatives of the connection and are linear in the second derivatives of the connection. Then I rederive the identities which guarantee the Noether conservation laws of energy-momentum and angular momentum. Following that I report on the status of Isenberg's and my work on finding an initial value formulation of the theory. We have divided the field equations into initial value constraints and evolution equations but have not yet completed the preservation of the constraints. Next, I repeat Fairchild's discussion of the Newtonian limit and give reasons why it is insufficient in its treatment of the torsion field. Hence, more work is needed on the Newtonian limit. Finally, I point out that the spatially flat Friedmann-Robertson-Walker solution of Einstein's theory with $p = \frac{1}{3}\rho$ and zero torsion is also an exact solution of this theory, thus providing an example of a cosmological solution.

In Section 3, I present a Birkhoff theorem proved by Ramaswamy and myself. It shows that the unique O(3)-spherically symmetric vacuum solution to this theory is the Schwarzschild metric and zero torsion. This shows that if the gravitational field of the solar system were precisely spherically symmetric (including reflection), then the theory with Lagrangian (1) would make the same predictions for solar system experiments as does Einstein's theory. A stronger statement about the predictions of this theory for solar system experiments can be made only after checking the stability of the Schwarzschild solution under perturbations involving torsion. The Birkhoff theorem also implies that Minkowski space with zero torsion is the locally unique vacuum solution which is spatially homogeneous, isotropic and parity-invariant. Thus Minkowski space serves as a ground state of the gravitational field.

2. Automatic Noether Conservation Laws, Initial Value Formulation, etc.

In the Lagrangian,

$$L = -s\,\frac{\hbar c}{16\pi L^2}\,\hat{R} - \frac{\hbar c}{16\pi\alpha_G}\,\hat{R}^{\alpha}{}_{\beta\gamma\delta}\,\hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta} + L_M, \tag{1}$$

$L = (\hbar G/c^3)^{1/2}$ is the Planck length, α_G is a new dimensionless coupling constant (which I regard as a gravitational fine structure constant), L_M is the matter Lagrangian minimally coupled to the Cartan connection, and the gravitational variables are chosen as the components of the orthonormal 1-form frame, $\theta^{\alpha}{}_{a}$, and the mixed components of the Cartan connection, $\Gamma^{\alpha}{}_{\beta a}$.

This Lagrangian (1) is the special case of the Lagrangian (V.3.109) in which

$$\Lambda = c_2 = a_1 = a_2 = a_4 = a_5 = a_6 = b_1 = b_2 = b_3 = 0,$$

$$c_1 = 1 \quad , \quad a_3 = 1/\alpha_G. \tag{2}$$

Hence, the field equations may be obtained from equations (V.3.112) and (V.3.113). The variation with respect to $\theta^{\alpha}{}_{a}$ yields the Einstein equation,

$$\hat{G}^{a}{}_{\alpha} - s\;2\chi(\hat{R}^{\kappa\lambda\mu a}\,\hat{R}_{\kappa\lambda\mu\alpha} - \frac{1}{4}\,e_{\alpha}{}^{a}\,\hat{R}^{\kappa\lambda\mu\nu}\,\hat{R}_{\kappa\lambda\mu\nu}) = \frac{8\pi L^2}{\hbar c}\;t_{\alpha}{}^{a}, \tag{3}$$

where $\chi = L^2/\alpha_G$. The variation with respect to $\Gamma^{\alpha}{}_{\beta a}$ yields the Cartan equation,

$$-2\,g^{\beta\delta}\,\nabla_b(e_{\alpha}{}^{[a}\,e_{\delta}{}^{b]}) - s\;4\chi\,\nabla_b\hat{R}^{\beta}{}_{\alpha}{}^{ab} = \frac{8\pi L^2}{\hbar c}\;S^{\beta}{}_{\alpha}{}^{a}. \tag{4}$$

In the vacuum, $L_M = 0$, $t_{\alpha}{}^{a} = 0$ and $S^{\beta}{}_{\alpha}{}^{a} = 0$.

It is easy to verify that any vacuum solution of the Einstein theory,

$$\tilde{G}^{a}{}_{\alpha} = 0, \tag{5}$$

is also a solution of equations (3) and (4) in vacuum with zero torsion. This shows that the theory based on Lagrangian (1) is consistent. What is more, Debney, Fairchild and Siklos have proved that in vacuum, in the absence of torsion, equations (3) and (4) are exactly equivalent to equation (5), i.e. they have the same solution sets. Note that Debney, Fairchild and Siklos take "vacuum" to mean torsion-free as well as no matter. I take it to mean only that there are no matter fields present.

By Theorem V.3, this theory has automatic Noether conservation laws. I first discuss the requisite identities. For ease of reference, it is useful to introduce the tensors

$$E^{1}{}_{\alpha}{}^{a} = \frac{\hbar c}{8\pi L^2}\, \hat{G}^{a}{}_{\alpha}, \tag{6}$$

$$E^{2}{}_{\alpha}{}^{a} = -\, s\, \frac{\hbar c}{4\pi\alpha_G}\, (\hat{R}^{\kappa\lambda\mu a}\, \hat{R}_{\kappa\lambda\mu\alpha} - \frac{1}{4}\, e_{\alpha}{}^{a}\, \hat{R}^{\kappa\lambda\mu\nu}\, \hat{R}_{\kappa\lambda\mu\nu}) \,, \tag{7}$$

$$C^{1\beta}{}_{\alpha}{}^{a} = -\, \frac{\hbar c}{4\pi L^2}\, g^{\beta\delta}\, \nabla_b (e_{\alpha}{}^{[a}\, e_{\delta}{}^{b]}) \tag{8}$$

$$= -\, \frac{\hbar c}{8\pi L^2}\, (\lambda^{a\beta}{}_{\alpha} - \lambda^{a}{}_{\alpha}{}^{\beta} + \lambda^{\beta\delta}{}_{\delta}\, e_{\alpha}{}^{a} - \lambda_{\alpha}{}^{\delta}{}_{\delta}\, g^{\beta\gamma}\, e_{\gamma}{}^{a}) \tag{9}$$

$$= -\, \frac{\hbar c}{8\pi L^2}\, (Q^{a}{}_{\alpha}{}^{\beta} - Q^{\delta}{}_{\delta}{}^{\beta}\, e_{\alpha}{}^{a} + Q^{\delta}{}_{\delta\alpha}\, g^{\beta\gamma}\, e_{\gamma}{}^{a}), \tag{10}$$

$$C^{2\beta}{}_{\alpha}{}^{a} = -\, s\, \frac{\hbar c}{2\pi\alpha_G}\, \nabla_b \hat{R}^{\beta}{}_{\alpha}{}^{ab} \,. \tag{11}$$

So the field equations read

$$E^{1}{}_{\alpha}{}^{a} + E^{2}{}_{\alpha}{}^{a} = t_{\alpha}{}^{a} , \tag{12}$$

$$C^{1\beta}{}_{\alpha}{}^{a} + C^{2\beta}{}_{\alpha}{}^{a} = S^{\beta}{}_{\alpha}{}^{a} . \tag{13}$$

To derive the energy-momentum conservation law, I begin with the Bianchi identity,

$$\eta^{abcd}\, \nabla_c \hat{R}^{\alpha}{}_{\beta ab} = 0, \tag{14}$$

which may also be written as

$$0 = \nabla_a \hat{R}^{\alpha\beta}{}_{bc} + \nabla_b \hat{R}^{\alpha\beta}{}_{ca} + \nabla_c \hat{R}^{\alpha\beta}{}_{ab}. \tag{15}$$

Contracting with $e_{\alpha}{}^{a}\ e_{\beta}{}^{b}\ e_{\gamma}{}^{c}$ yields

$$\begin{aligned} 0 &= -\nabla_a \hat{R}^{a}{}_{\gamma} - \nabla_b \hat{R}^{b}{}_{\gamma} + \nabla_c (e_{\gamma}{}^{c} \hat{R}) \\ &\quad - [\nabla_a (e_{\alpha}{}^{a}\ e_{\beta}{}^{b})] \hat{R}^{\alpha\beta}{}_{b\gamma} - [\nabla_b (e_{\alpha}{}^{a}\ e_{\beta}{}^{b})] \hat{R}^{\alpha\beta}{}_{\gamma a} - [\nabla_c (e_{\alpha}{}^{a}\ e_{\beta}{}^{b})] e_{\gamma}{}^{c}\ \hat{R}^{\alpha\beta}{}_{ab} \\ &\quad + (\nabla_a e_{\gamma}{}^{c}) \hat{R}^{a}{}_{c} + (\nabla_b e_{\gamma}{}^{c}) \hat{R}^{b}{}_{c} - (\nabla_c e_{\gamma}{}^{c}) \hat{R} \\ &= -2\, \nabla_a \hat{G}^{a}{}_{\gamma} - 2[\nabla_b (e_{\alpha}{}^{[a}\ e_{\beta}{}^{b]})] \hat{R}^{\alpha\beta}{}_{\gamma a} + 2\ \hat{G}^{a}{}_{c}\ Q^{c}{}_{\gamma a}, \end{aligned} \tag{16}$$

where I have used equation (V.3.114). Thus,

$$\nabla_a E^{1}{}_{\gamma}{}^{a} = \frac{1}{2}\, C^{1\beta}{}_{\alpha}{}^{\delta}\ \hat{R}^{\alpha}{}_{\beta\gamma\delta} + E^{1}{}_{\alpha}{}^{\delta}\ Q^{\alpha}{}_{\gamma\delta}. \tag{17}$$

Next by direct computation

$$\nabla_a E^{2\ a}_{\ \gamma} = - s \frac{\hbar c}{4\pi\alpha_G} [(\nabla_a \hat{R}^{\kappa\lambda ba})\hat{R}_{\kappa\lambda b\gamma}$$

$$+ \hat{R}^{\kappa\lambda ba}(\nabla_a e_\gamma{}^c)\hat{R}_{\kappa\lambda bc} + \hat{R}^{\kappa\lambda ba}\ e_\gamma{}^c\ \nabla_a \hat{R}_{\kappa\lambda bc}$$

$$- \frac{1}{4}(\nabla_a e_\gamma{}^a)\hat{R}^{\kappa\lambda bd}\ \hat{R}_{\kappa\lambda bd} - \frac{1}{2}\ e_\gamma{}^c\ \hat{R}^{\kappa\lambda ba}\ \nabla_c \hat{R}_{\kappa\lambda ba}]$$

$$= - s \frac{\hbar c}{4\pi\alpha_G} [(\nabla_a \hat{R}^\beta{}_\alpha{}^{ba})\hat{R}^\alpha{}_{\beta\gamma b}$$

$$+ (\hat{R}^{\kappa\lambda ba}\ \hat{R}_{\kappa\lambda bc} - \frac{1}{4}\ \delta^a_c\ \hat{R}^{\kappa\lambda bd}\ \hat{R}_{\kappa\lambda bd})\nabla_a e_\gamma{}^c]$$

$$= \frac{1}{2}\ C^{2\beta}{}_\alpha{}^\delta\ \hat{R}^\alpha{}_{\beta\gamma\delta} + E^{2}{}_\alpha{}^\delta\ Q^\alpha{}_{\gamma\delta}, \tag{18}$$

where the second step uses the Bianchi identity (15) and the third step uses

$$Q_{c\gamma a} + Q_{a\gamma c} = - \lambda_{c\gamma a} - \lambda_{a\gamma c} = g_{cd}\ \nabla_a e_\gamma{}^d + g_{ad}\nabla_c e_\gamma{}^d. \tag{19}$$

The identities (17) and (18) together with the field equations (12) and (13) guarantee the conservation of energy-momentum:

$$\nabla_a t_\gamma{}^a = \frac{1}{2}\ S^\beta{}_\alpha{}^\delta\ \hat{R}^\alpha{}_{\beta\gamma\delta} + t_\alpha{}^\delta\ Q^\alpha{}_{\gamma\delta}. \tag{20}$$

To derive the angular momentum conservation law, I use the Ricci identity to compute

$$\nabla_a\nabla_b(e_\alpha{}^{[a}\ e_\beta{}^{b]}) = \frac{1}{2}\ (\tilde{R}^a{}_{cab}\ e_\alpha{}^c\ e_\beta{}^b + \tilde{R}^b{}_{cab}\ e_\alpha{}^a\ e_\beta{}^c$$

$$- \hat{R}^\gamma{}_{\alpha ab}\ e_\gamma{}^a\ e_\beta{}^b - \hat{R}^\gamma{}_{\beta ab}\ e_\alpha{}^a\ e_\gamma{}^b)$$

$$= - \frac{1}{2}\ (\hat{R}_{\alpha\beta} - \hat{R}_{\beta\alpha}) = - \frac{1}{2}\ (\hat{G}_{\alpha\beta} - \hat{G}_{\beta\alpha}), \tag{21}$$

or

$$\nabla_a C^{1}{}_{\beta\alpha}{}^{a} = E^{1}{}_{\beta\alpha} - E^{1}{}_{\alpha\beta}. \tag{22}$$

On the other hand, again using the Ricci identity,

$$\nabla_a \nabla_b \hat{R}_{\beta\alpha}{}^{ab} = \frac{1}{2} (\tilde{R}^{a}{}_{cab}\ \hat{R}_{\beta\alpha}{}^{cb} + \tilde{R}^{b}{}_{cab}\ \hat{R}_{\beta\alpha}{}^{ac}$$

$$- \hat{R}^{\gamma}{}_{\beta ab}\ \hat{R}_{\gamma\alpha}{}^{ab} - \hat{R}^{\gamma}{}_{\alpha ab}\ \hat{R}_{\beta\gamma}{}^{ab})$$

$$= 0 , \tag{23}$$

or

$$\nabla_a C^{2}{}_{\beta\alpha}{}^{a} = 0 = E^{2}{}_{\beta\alpha} - E^{2}{}_{\alpha\beta}. \tag{24}$$

The identities (22) and (24) together with the field equations (12) and (13) guarantee the conservation of angular momentum:

$$\nabla_a S_{\beta\alpha}{}^{a} = t_{\beta\alpha} - t_{\alpha\beta}. \tag{25}$$

It may be informative to interpret the tensor,

$$\tau_{\alpha}{}^{a} = - E^{2}{}_{\alpha}{}^{a} , \tag{26}$$

as an energy-momentum tensor for $\Gamma^{\alpha}{}_{\beta a}$, and to interpret the tensor,

$$\sigma^{\beta}{}_{\alpha}{}^{a} = - C^{1\beta}{}_{\alpha}{}^{a} , \tag{27}$$

as a spin tensor for $\theta^{\alpha}{}_{a}$. From this point of view the field equations may be written as

$$\hat{G}^{a}{}_{\alpha} = \frac{8\pi L^2}{\hbar c} \ (t_{\alpha}{}^{a} + \tau_{\alpha}{}^{a}) , \tag{28}$$

$$\nabla_b R^{\beta}{}_{\alpha}{}^{ab} = - s \frac{2\pi\alpha_G}{\hbar c} (S^{\beta}{}_{\alpha}{}^{a} + \sigma^{\beta}{}_{\alpha}{}^{a}). \qquad (29)$$

Equation (28) is the field equation for $\theta^{\alpha}{}_{a}$ while (29) is the field equation for $\Gamma^{\alpha}{}_{\beta a}$.

Alternatively, it may be useful to interpret $\sigma^{\beta}{}_{\alpha}{}^{a}$ as a tensorial geometric manifestation of the orbital angular momentum, since from equations (29) and (23) it follows that,

$$\nabla_a (S^{\beta}{}_{\alpha}{}^{a} + \sigma^{\beta}{}_{\alpha}{}^{a}) = 0 \; . \qquad (30)$$

Thus spin plus σ are conserved.

Isenberg and I have begun to investigate the initial value formulation of this theory in the vacuum. By inspection, the vacuum Lagrangian (equation (1) with $L_M = 0$) and the vacuum Einstein equations (equation (3) with $t_{\alpha}{}^{a} = 0$) are both strictly local functions of $\theta^{\alpha}{}_{a}$, $\Gamma^{\alpha}{}_{\beta a}$, and $\partial_b \Gamma^{\alpha}{}_{\beta a}$; while the vacuum Cartan equations (equation (4) with $S^{\beta}{}_{\alpha}{}^{a} = 0$) are strictly local functions of $\theta^{\alpha}{}_{a}$, $\partial_b \theta^{\alpha}{}_{a}$, $\Gamma^{\alpha}{}_{\beta a}$, $\partial_b \Gamma^{\alpha}{}_{\beta a}$, and $\partial_c \partial_b \Gamma^{\alpha}{}_{\beta a}$ which are linear in $\partial_c \partial_b \Gamma^{\alpha}{}_{\beta a}$. Since the Lagrangian contains no derivatives of $\theta^{\alpha}{}_{a}$, the components of the frame, $\theta^{\alpha}{}_{a}$, play the role of Lagrange multipliers, and all 16 of the Einstein equations,

$$E_{\alpha}{}^{a} = E^{1}{}_{\alpha}{}^{a} + E^{2}{}_{\alpha}{}^{a} = 0, \qquad (31)$$

are initial value constraints.

Further examination of the Lagrangian shows that it depends on the connection only through the curvature. Hence the only components of the connection which have time derivatives in the Lagrangian are $\Gamma^{\alpha}{}_{\beta a}$ for $a = 1,2,3$. As a consequence, these are the only gravitational variables which have conjugate momenta in a canonical formulation. They are also the only gravitational variables which have second time derivatives in the Cartan equations,

$$C^{\beta}{}_{\alpha}{}^{a} = C^{1\beta}{}_{\alpha}{}^{a} + C^{2\beta}{}_{\alpha}{}^{a} = 0. \tag{32}$$

Their dynamic equations are the Cartan equations,

$$C^{\beta}{}_{\alpha}{}^{a} = 0 \text{ with } a = 1,2,3. \tag{33}$$

The remaining Cartan equations,

$$C^{\beta}{}_{\alpha}{}^{o} = 0, \tag{34}$$

are initial value constraints.

To complete the verification that the vacuum theory has a good initial value formulation, it remains to check that the constraints (31) and (34) are preserved in time. The Cartan constraints (34) are preserved by virtue of the identity,

$$\nabla_a C_{\beta\alpha}{}^{a} = E_{\beta\alpha} - E_{\alpha\beta} , \tag{35}$$

which follows from (22) and (24). The Einstein constraints (31) with a=0 are preserved by virture of the identity,

$$\nabla_a E_{\gamma}{}^{a} = \frac{1}{2} C^{\beta}{}_{\alpha}{}^{a} \hat{R}^{\alpha}{}_{\beta\gamma a} + E_{\alpha}{}^{a} Q^{\alpha}{}_{\gamma a}, \tag{36}$$

which follows from (17) and (18). Examination of the space-space components of the Einstein constraints (equation (31) with $\alpha = 1,2,3$ and $a = 1,2,3$) shows that in principle, in a generic situation, it should be possible to solve these 9 constraints for the 9 space-space components of the frame, $\vartheta^{\alpha}{}_{a}$ with $\alpha = 1,2,3$ and $a = 1,2,3$. In fact, Isenberg and I have shown that this is possible in vacuum for the metric sufficiently close to

Minkowski and the torsion sufficiently small. The ability to solve a constraint for a variable guarantees that the constraint is preserved. Isenberg and I are presently in the process of checking that the remaining 3 Einstein constraints (equation (31) with $\alpha = 0$ and $a = 1,2,3$) are preserved.

In his paper, Fairchild claims to check the Newtonian limit of the theory with Lagrangian (1). He argues as follows: The Einstein equations (28) may be written as

$$\tilde{G}_{\beta\alpha} = \frac{8\pi L^2}{\hbar c}\ (t_{\alpha\beta} + \tau_{\alpha\beta} + V_{\alpha\beta}), \tag{37}$$

where

$$\frac{8\pi L^2}{\hbar c}\ V_{\alpha\beta} = \tilde{G}_{\beta\alpha} - \hat{G}_{\beta\alpha}$$
$$= e_\alpha{}^a(\nabla_a\lambda^c{}_{\beta c} - \nabla_c\lambda^c{}_{\beta a}) - \frac{1}{2}\, g_{\alpha\beta}\, e_\gamma{}^a(\nabla_a\lambda^{c\gamma}{}_c - \nabla_c\lambda^{c\gamma}{}_a). \tag{38}$$

Then $\tau_{\alpha\beta}$ may be neglected since it is quadratic in the curvature and hence second order in the metric and torsion perturbations away from torsion-free Minkowski space. If $t_{oo} + V_{oo}$ is identified as the mass density then the Newtonian limit follows as in Einstein's theory.

The first problem with this argument is that $t_{oo} + V_{oo}$ is not the mass density; t_{oo} is! However, the argument could be salvaged if it could be shown that the torsion can be chosen sufficiently small so that $V_{\alpha\beta}$ is negligible compared to $t_{\alpha\beta}$. The second problem is that the argument does not mention the Cartan equation. One must check that orders in v/c can be consistently assigned to each component of the metric and torsion perturbations so that not only does the G_{oo} equation reduce to Newton's equation but also the remaining Einstein equations and all of the Cartan equations can be satisfied to lowest order. It may not be possible simultaneously to make the torsion sufficiently small so that $V_{\alpha\beta}$ is negligible compared to $t_{\alpha\beta}$ and also to satisfy the Cartan equations. This is particularly crucial because when regarded as a function of the metric and torsion, the Cartan equation contains third derivatives of the metric. I believe this theory will have a good Newtonian limit, but it is not proven. More work is needed.

Finally, I demonstrate that there is at least one cosmological solution, namely the radiation dominated ($p = \frac{1}{3}\rho$), spatially flat, Friedmann solution to Einstein's theory with zero torsion. When the torsion is zero, the field equations (3) and (4) reduce to

$$\tilde{G}_{ab} + s\ 4\chi\ \tilde{C}_{acbd}\ \tilde{R}^{cd} = \frac{8\pi L^2}{\hbar c}\ t_{ab}\ , \tag{39}$$

$$\nabla_b \tilde{R}_{cd}{}^{ab} = -\ s\ \frac{2\pi\alpha_G}{\hbar c}\ S_{cd}{}^a\ . \tag{40}$$

For a 4-dimensionally conformally flat metric, such as the Friedmann-Robertson-Walker metric,

$$ds^2 = a^2(\eta)\ (-d\eta^2 + dr^2 + r^2\ d\theta^2 + r^2\ \sin^2\theta\ d\phi^2)\ , \tag{41}$$

the Weyl tensor C_{acbd} is zero. Hence the Einstein equation (39) becomes

$$\tilde{G}_{ab} = \frac{8\pi L^2}{\hbar c}\ t_{ab}\ , \tag{42}$$

which is satisfied by any 4-dimensionally conformally flat solution of Einstein's theory. But the Cartan equation (40) must also be satisfied. Using a contracted Bianchi identity and then (42) it can be rewritten as

$$-\ s\ \frac{2\pi\alpha_G}{\hbar c}\ S_{cd}{}^a = \nabla_d \tilde{R}_c{}^a - \nabla_c \tilde{R}_d{}^a = \frac{8\pi L^2}{\hbar c}\ (\nabla_d \bar{t}_c{}^a - \nabla_c \bar{t}_d{}^a)\ , \tag{43}$$

where

$$\bar{t}_c{}^a = t_c{}^a - \frac{1}{2}\delta_c{}^a t_b{}^b . \tag{44}$$

This may be interpreted as a definition of the spin tensor or as a restriction on the energy-momentum tensor. If the spin tensor is set to zero, and the energy-momentum tensor is chosen as that for a perfect fluid, then the restriction (43) reduces to

$$p = \frac{1}{3}\rho + \text{constant}. \tag{45}$$

Thus the radiation dominated, spatially flat, Friedmann cosmology with zero torsion satisfies the field equations of Lagrangian (1).

3. Birkhoff Theorem

Sriram Ramaswamy and I have proved a Birkhoff theorem for the vacuum Lagrangian,

$$L = - s \frac{\hbar c}{16\pi L^2} \hat{R} - \frac{\hbar c}{16\pi\alpha_G} \hat{R}^{\alpha}{}_{\beta\gamma\delta} \hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}. \tag{1}$$

The theorem states,

Theorem VI.1 (Birkhoff Theorem):

Let the metric and Cartan connection on a region of spacetime be an 0(3)-spherically symmetric vacuum solution to the field equations derived from Lagrangian (1). Then the connection has zero torsion, and the metric is the Schwarzschild metric.

This theorem is analogous to Birkhoff's theorem for Einstein's theory. Before giving the proof, I discuss its implications.

First, this theorem should be compared with a result by Debney, Fairchild and Siklos (DFS). They have proved that in vacuum, in the absence of torsion, this theory is exactly equivalent to Einstein's theory in vacuum; i.e. the two restricted theories have the same solution sets. (Note that these authors take "vacuum" to mean torsion-free as well as matter-free. I take " vacuum" to mean only that there is no matter present.) In contrast, in our Birkhoff theorem, Ramaswamy and I do not assume that the torsion is zero. Instead, we assume 0(3)-spherical symmetry and prove that the torsion is zero. Then the standard Birkhoff theorem for Eisntein's theory, in conjunction with the DFS result, shows that the metric is Schwarzschild. In our proof we do not actually quote the DFS result; we rederive it in the special case of 0(3)-spherical symmetry.

Second, unfortunately there is at present no verification that (1) this theory has a good Newtonian limit, and (2) the Schwarzschild solution is stable under non-spherical perturbations. If there were, then as discussed in Section V.2.b, our Birkhoff theorem would show that this theory makes the same predictions about solar system experiments as does Einstein's theory. Since the Schwarzschild solution is stable in Einstein's theory, the DFS result shows that any non-spherical perturbations which are unstable in this theory must involve a non-zero torsion field.

Third, I want to emphasize that our Birkhoff theorem assumes O(3)-spherical symmetry rather than just SO(3)-spherical symmetry; i.e. spatial reflections as well as rotation. This means that there is an action of the group O(3) on the spacetime manifold which leaves the metric and torsion invariant and whose orbits are generically 2-spheres. For the metric and any other symmetric rank two tensor, the assumptions of O(3) and SO(3) symmetry are equivalent. However, for the torsion, the inclusion of reflections as well as rotations reduces the number of torsion functions from eight to four in equations (10-13) below.

In fact, when the spherical symmetry assumption is weakened from O(3) to SO(3), the local uniqueness of the torsion-free Schwarzschild solution breaks down in that there exist local SO(3)-spherically symmetric solutions whose torsion is not space reversal invariant. I am able to demonstrate the existence of a five parameter family of such solutions which are also static (in the technical sense that the timelike killing vector is surface orthogonal) but not time reversal invariant. This is done by (1) writing out the static, SO(3)-spherically symmetric field equations, (2) proving that four of the eight torsion components vanish, and (3) checking that the resultant equations have an initial value formulation in the radial direction whose initial conditions are the values of the four remaining torsion components and the value of one metric function (which fixes the Schwarzschild mass when the torsion is zero). This proves the existence of a five parameter family of solutions in the neighborhood of the initial radial coordinate. I do not have any closed form expression for these solutions and I have not yet checked the asymptotic form of these fields to see if they are asymptotically flat.

The existence of these parity non-invariant solutions can be understood by considering the analogy with the Birkhoff theorem for the Einstein-Maxwell theory. In that theory the unique O(3)-spherically symmetric solution is the Reissner-Nordstrom metric with a radial electric field. When the spherical symmetry assumption is relaxed from O(3) to SO(3), radial magnetic fields are also permitted.

If one regards Birkhoff's theorem as merely saying that every spherically symmetric solution is static, then perhaps a Birkhoff theorem for the theory based on Lagrangian (1) may still exist for SO(3)-spherical symmetry.

Finally, as pointed out in Section V.3.f our Birkhoff theorem implies that Minkowski space with zero torsion is the locally unique vacuum solution which is spatially homogeneous, isotropic and parity-invariant. This follows because such a solution would have to be O(3)-spherically symmetric about every point. Thus Minkowski space serves as the ground state of the gravitational field in this theory.

I conclude this section and the thesis with the proof of the Birkhoff theorem proven by Ramaswamy and myself.

Proof of Theorem VI.1 (Birkhoff Theorem):

We first give an outline of our proof. After writing out the field equations and Bianchi identities for a spherically symmetric system, we note that the Einstein equations factor, yielding three cases. In two of the cases, adding the Bianchi identities to the Cartan equations leads to a contradiction. In the third case, subtracting the Bianchi identities from the Cartan equations shows that the Einstein and torsion tensors are zero. Birkhoff's theorem for Einstein's theory then implies that the Schwarzschild metric is the unique solution. The details of the proof follow.

In this proof, we take the signature of the metric to be (-1,+1,+1,+1). Recall that the Bianchi identities are

$$\eta^{abcd}\,\nabla_c \hat{R}_{\beta\alpha ab} = 0, \tag{2}$$

the vacuum Einstein equations are

$$\hat{G}_{\mu\nu} - 2\chi(\hat{R}^{\alpha}{}_{\beta\gamma\mu}\,\hat{R}^{\beta}{}_{\alpha}{}^{\gamma}{}_{\nu} - \frac{1}{4}\,g_{\mu\nu}\,\hat{R}^{\alpha}{}_{\beta\gamma\delta}\hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}) = 0\ , \tag{3}$$

and the vacuum Cartan equations are

$$\lambda^{c}{}_{\alpha\beta} - \lambda^{c}{}_{\beta\alpha} + e_{\alpha}{}^{c}\,\lambda^{\delta}{}_{\beta\delta} - e_{\beta}{}^{c}\,\lambda^{\delta}{}_{\alpha\delta} + 4\chi\,\nabla_d\hat{R}_{\beta\alpha}{}^{cd} = 0, \tag{4}$$

where $\chi = L^2/\alpha_G$.

The most general spherically symmetric metric can be written as

$$ds^2 = -\,e^{2\Phi}\,dT^2 + e^{2\Lambda}\,dR^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\ , \tag{5}$$

where Φ, Λ and r are arbitrary functions of R and T. We are allowing here for the possibility that the Schwarzschild r is a bad coordinate, by making r an arbitrary function of good coordinates R and T. We write out all tensors in the orthonormal frame, with basis one-forms,

$$\theta^T = e^{\Phi}\, dT \ , \tag{6}$$

$$\theta^R = e^{\Lambda}\, dR \ , \tag{7}$$

$$\theta^\theta = r\, d\theta \ , \tag{8}$$

$$\theta^\phi = r \sin\theta\, d\phi \ . \tag{9}$$

We impose O(3)-spherical symmetry on the Cartan connection by demanding that the defect tensor be invariant under rotations and reflections, and that the Christoffel symbols be computed from the spherically symmetric metric (5). The independent non-zero components of the spherically symmetric defect tensor are

$$\lambda^T{}_{RT} = f(R,T) \ , \tag{10}$$

$$\lambda^T{}_{RR} = h(R,T) \ , \tag{11}$$

$$\lambda^T{}_{\theta\theta} = \lambda^T{}_{\phi\phi} = k(R,T) \ , \tag{12}$$

$$\lambda^R{}_{\theta\theta} = \lambda^R{}_{\phi\phi} = g(R,T) \ , \tag{13}$$

where f, g, h, k are arbitrary functions of R and T. Adding the defect tensor to the Christoffel symbols, we obtain the independent non-zero components of the Cartan connection:

$$\Gamma^T{}_{RT} = e^{-\Lambda}\Phi' + f \qquad \equiv \ V(R,T) \ , \tag{14}$$

$$\Gamma^T{}_{RR} = e^{-\Phi}\,\dot{\Lambda} + h \qquad \equiv \ X(R,T) \ , \tag{15}$$

$$\Gamma^T{}_{\theta\theta} = \Gamma^T{}_{\phi\phi} = e^{-\Phi} r^{-1}\,\dot{r} + k \ \equiv \ Y(R,T) \ , \tag{16}$$

$$\Gamma^R{}_{\theta\theta} = \Gamma^R{}_{\phi\phi} = -e^{-\Lambda} r^{-1}\, r' + g \ \equiv \ W(R,T) \ , \tag{17}$$

$$\Gamma^{\theta}{}_{\phi\phi} = - r^{-1} \cot\theta \, . \tag{18}$$

Dots (·) denote differentiation with respect to T, and primes (') denote differentiation with respect to R.

The independent non-zero components of the Riemann tensor are

$$\hat{R}^{T}{}_{RTR} = [(Xe^{\Lambda})^{\cdot} - (Ve^{\Phi})']e^{-\Phi-\Lambda} \equiv -A \, , \tag{19}$$

$$\hat{R}^{T}{}_{\theta T\theta} = \hat{R}^{T}{}_{\phi T\phi} = e^{-\Phi}r^{-1}(Yr)^{\cdot} + VW \equiv -C \, , \tag{20}$$

$$\hat{R}^{T}{}_{\theta R\theta} = \hat{R}^{T}{}_{\phi R\phi} = e^{-\Lambda}r^{-1}(Yr)' + XW \equiv D \, , \tag{21}$$

$$\hat{R}^{R}{}_{\theta T\theta} = \hat{R}^{R}{}_{\phi T\phi} = e^{-\Phi}r^{-1}(Wr)^{\cdot} + YV \equiv -G \, , \tag{22}$$

$$\hat{R}^{R}{}_{\theta R\theta} = \hat{R}^{R}{}_{\phi R\phi} = e^{-\Lambda}r^{-1}(Wr)' + YX \equiv H \, , \tag{23}$$

$$\hat{R}^{\theta}{}_{\phi\theta\phi} = r^{-2} + Y^{2} - W^{2} \equiv L \, , \tag{24}$$

and the non-zero components of the Einstein tensor are

$$\hat{G}_{TT} = 2H + L \, , \tag{25}$$

$$\hat{G}_{TR} = - 2D \, , \tag{26}$$

$$\hat{G}_{RT} = - 2G \, , \tag{27}$$

$$\hat{G}_{RR} = 2C-L \, , \tag{28}$$

$$\hat{G}_{\theta\theta} = \hat{G}_{\phi\phi} = C-H+A \, . \tag{29}$$

The independent Einstein equations (3) in vacuum are

$$2H+L - 4\chi(D^2+C^2-H^2-G^2- \tfrac{1}{2} L^2 + \tfrac{1}{2} A^2) = 0, \qquad (30)$$

$$-2D + 8\chi(CD-HG) = 0, \qquad (31)$$

$$-2G + 8\chi(CD-HG) = 0, \qquad (32)$$

$$2C - L - 4\chi(D^2+C^2-H^2-G^2+ \tfrac{1}{2} L^2 - \tfrac{1}{2} A^2) = 0, \qquad (33)$$

$$C-H+A+2\chi(L^2-A^2) = 0. \qquad (34)$$

The independent Cartan equations (4) in vacuum are

$$-2(W+e^{-\Lambda}r^{-1}r') + 4\chi[e^{-\Lambda}r^{-2}(r^2A)' - 2\,YG + 2\,WC] = 0, \qquad (35)$$

$$-2(Y-e^{-\Phi}r^{-1}\dot{r}) - 4\chi[e^{-\Phi}r^{-2}(r^2A)^{\cdot} + 2\,YH-2WD] = 0, \qquad (36)$$

$$-[X+Y-e^{-\Phi-\Lambda}r^{-1}(re^{\Lambda})^{\cdot}] - 4\chi\{e^{-\Phi-\Lambda}r^{-1}[(Dre^{\Phi})' + (Cre^{\Lambda})^{\cdot}]$$

$$+ VG + XH + YL\} = 0, \qquad (37)$$

$$-[W-V+e^{-\Phi-\Lambda}r^{-1}(re^{\Phi})'] - 4\chi\{e^{-\Phi-\Lambda}r^{-1}[(Hre^{\Phi})' + (Gre^{\Lambda})^{\cdot}]$$

$$+ VC + XD + WL\} = 0. \qquad (38)$$

The independent Bianchi identities (2) are

$$- e^{-\Lambda}r^{-2}(r^2L)' + 2\ YD - 2\ WH = 0\ , \tag{39}$$

$$e^{-\Phi}r^{-2}(r^2L)^{\cdot} + 2\ YC - 2\ WG = 0, \tag{40}$$

$$e^{-\Phi-\Lambda}r^{-1}[(Gre^{\Phi})' + (Hre^{\Lambda})^{\cdot}] + VD + XC + YA = 0, \tag{41}$$

$$e^{-\Phi-\Lambda}r^{-1}[(Cre^{\Phi})' + (Dre^{\Lambda})^{\cdot}] + VH + XG + WA = 0. \tag{42}$$

Based upon equations (30-42), we now prove that the defect and Einstein tensors are zero. First, the Einstein equations (30-34) can be manipulated into the equivalent form:

$$G = D, \tag{43}$$

$$G[1 - 4\chi(C-H)] = 0\ , \tag{44}$$

$$(H + C)\ [1 - 4\chi(C - H)] = 0, \tag{45}$$

$$(A - L) + 2\ (C - H) = 0\ , \tag{46}$$

$$(A + L)[1 + 8\chi(C - H)] = 0\ . \tag{47}$$

These equations split into three cases ,

Case I: $C\text{-}H = (4\chi)^{-1}$, so that $G=D$ and $L=-A=(4\chi)^{-1}$.

Case II: $C\text{-}H = -\ (8\chi)^{-1}$, so that $G=D=0$, $H=-C=(16\chi)^{-1}$, and $A\text{-}L = (4\chi)^{-1}$.

Case III: $C\text{-}H \neq -\ (8\chi)^{-1}$ and $C\text{-}H \neq (4\chi)^{-1}$, so that $G=D=0$, and $A=-L=-2C=2H$.

Next, we compare the Cartan equations (35-38) with the Bianchi identities (39-42) in each of the three cases.

Case I: We add 4χ times (41) to (37), and 4χ times (42) to (38), and use the conditions of case I on the resulting two equations to show that Y=W=0. From the definitions (20) and (23) of C and H, this implies C=H=0, which contradicts $C-H=(4\chi)^{-1}$, ruling out case I.

Case II: We add 4χ times (39) to (35), and 4χ times (40) to (36), and use the conditions of case II to show that Y=W=0. As before, this implies C=H=0, which contradicts $C-H=-(8\chi)^{-1}$, ruling out case II.

Case III: We subtract 4χ times (39) from (35), 4χ times (40) from (36), 4χ times (41) from (37), and 4χ times (42) from (38), and use the conditions of case III as well as equations (14-17) to show that f=g=h=k=0. This implies that the defect and hence the torsion tensors are zero. The conditions of case III directly imply that G=D =2H+L = 2C-L = C-H+A = 0. This implies that the Einstein tensor, computed with torsion, is zero, but since the torsion is zero, the Einstein tensor computed from the Christoffel connection is also zero. Birkhoff's theorem for Einstein's theory then says that the metric is the Schwarzschild metric. This is, in fact, a solution since with zero torsion, any solution to the vacuum field equations of Einstein's theory is a solution to equations (3) and (4) in vacuum. Hence the Schwarzschild metric, with zero torsion, is the unique solution.

This completes the proof of our generalized Birkhoff theorem.

Q.E.D.